# FINAL TECHNICAL REPORT

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"RESEARCH ON DETERMINISTIC AND STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO CONTINUUM PHYSICS AND STOCHASTIC SYSTEMS MODELLING."

Principal Investigator: Wendell H. Fleming

This is a summary of research completed during the period of this contract, 1 July 1983 - 30 June 1988. The work was done by four senior investigators in the Division of Applied Mathematics, Brown University ( C.M. Dafermos, W.H. Fleming, H.J. Kushner, and P.E. Souganidis) together with associated postdoctoral and graduate student personnel.

This report is arranged as follows. The work of Dafermos, Fleming, Kushner and Souganidis is summarized in turn, with references to research publications supported under this contract. Then a brief statement about other activities by these investigators and honors is appended.

## C.M. Dafermos

This research focused on a number of problems that lie at the interface of continuum mechanics and analysis. From the standpoint of analysis the mathematical issues concern existence, regularity, stability and large time behavior of solutions of nonlinear evolutionary partial differential equations, or systems thereof, of hyperbolic or mixed hyperbolic-parabolic type. From the viewpoint of continuum physics, the objective is to illuminate the stabilizing effects of such dissipation mechanisms as viscosity, heat diffusion and the second law of thermodynamics.

A major focus of Dafermos's work was the study of conservation laws (see references [DI] [D2] [D7 -D11].) Applications in solid mechanics include energy methods for problems in elastodynamics [D3] and studies of dissipation of energy and development of singularities in materials with memory [D4] [D5]. A feature of Dafermos's work is the consideration of continuum mechanical problems with nonconvex entropies, in which the structure of shocks for convex entropies fails to hold. Another application is to thermoelastic materials, including the effects of thermal softening [D6]. Yet another direction concerns two-phase flows [D12].



### Research Publications - C.M. Dafermos

[D1] - [D2]

D 1. "Large Time Behavior of Solutions of Hyperbolic Balance Laws." *Bull. Greek Math. Soc, .*, 25 (1984), 15.29.

D 2. "Regularity and Large Time Behavior of Solutions of a Conservation Law Without Convexity," *Proc. Royal Soc. Edinburgh*, 99A (1985), 201-239.

D 3. "Energy Methods for Quasilinear Hyperbolic Initial-boundary Value Problems: Applications to Elastodynamics,"( with W.J. Hrusa), *Arch. Rational Mech. Analysis*, 87 (1985), 267-292.

D 4. "Dissipation in Materials with Memory," *Viscoelasticity and Rheology* (A. Lodge, J.A. Nohel, and M. Renardy Eds.), Academic Press, New York (1985), 221-234.

D 5. "Development of Singularities in the Motion of Materials with Fading Memory," *Arch. Rational Mech. Analysis,* 91 (1986), 193-205.

D 6. "Development of Singularities in Solutions of the Equations of Nonlinear Thermoelasticity,"( with L. Hsiao), *Quart. Appl. Math.k* XLIV (1986), 463-474.

D 7. "Quasilinear Hyperbolic Systems with Involutions," *Arch. Rational Mech. Analysis*, 94 (1986), 373-389.

D 8. "Estimates for Conservation Laws with Little Viscosity," *SIAM J. Math. Analysis,* 18 (1987), 409-421.

D 9. "Trend to Steady State in a Conservation Law with Spatial Inhomogeneity," *Quart. Appl. math.,* XLV (1987), 313-319.

D 10. "Solutions in $L^\infty$ for a Conservation Law with Memory," *Analyses Mathematique et Applications,* Gauthier-Villars, Paris (1988), 117-128.

D 11. "Solutions with Shocks for Conservation Laws with Memory," *Amorphous Polymers and Non-Newtonian Fluids* (C. Dafermos, J.L. Ericksen, and D. Kinderlehrer, Eds.) Springer-Verlag, New York (1987,) 33-55.

D 12. . "Trajectories and Singular Points in Steady-State Models of two-Phase Flows"(with Z. Bilicki, J. Kestin, G. Majda, and D.L. Zeng), *Int. J. Multiphase Flows,* 13 (1987), 511-533.

W. H. Fleming

Fleming's research was concerned with optimal stochastic control theory, nonlinear filtering, large deviations for Markov diffusions, and viscosity solutions of nonlinear partial differential equations. The latter topic was studied jointly with Souganidis.

Fleming's work in optimal stochastic control focused on control of Markov diffusion processes, with complete or partial observations. This work is reported in references [F1] [F6] [F8] [F14 - F16]. Reference [F1] concerns basic mathematical results about controlled diffusion processes with partial observations, providing results about the existence of optimal relaxed controls and the nonlinear semigroup associated with the principle of dynamic programming. [F6] provides a solution to a stochastic production planning problem, making use of viscosity solution methods for a system of first-order nonlinear partial differential equations. [F7] [F14] provide an approximate solution, in the form of an asymptotic series in powers of a small parameter measuring the intensity of noise entering the control dynamics. In the context of filtering, [F12] also obtains an approximate solution in the form of a power series expansion. References [F8] [F16] provide an alternative to the usual dynamic programming, seeking maximal smooth subsolutions rather than solutions to the dynamic programming equations and using an abstract duality theorem from convex analysis.

The theory of large deviations is concerned with asymptotic estimates for exponentially small probabilities of rare events associated with stochastic processes. In 1977, Fleming introduced a stochastic control approach to large deviations of nearly deterministic Markov diffusions. This technique was based on a certain logarithmic transformation [F5]. Subsequently, simpler analytical methods based on logarithmic transformations and viscosity solution techniques were developed [F3] [F13]. These methods also give more accurate approximations in the form of an asymptotic series [F7][F14]. These analytical methods were adapted to other classes of Markov processes in [F11].

Fleming and Souganidis [F10][F17] developed a theory of value for two-player, zero sum stochastic differential games. This provides an optimization formula for viscosity solutions to a large class of nonlinear second partial differential equations, of degenerate parabolic type.

Research Publications W.H. Fleming

[F1] - [F17]

F 1. On Stochastic Relaxed Controls for Partially Observed Diffusions (with M. Nisio), Osaka Math. J.93 (1984) 71-108.

F 2. Optimal Control of Markov Processes, Proc. Intl. Congress of Mathematicians 1983 (invited Plenary Address).

F 3. A PDE Approach to Asymptotic Estimates for Optimal Exit Probabilities (with P.E. Souganidis), Annali della Scuola Normale Superiore Pisa, Ser. IV 23 (1986) 171-192.

F 4. Stochastic Variational Formula for Fundamental Solutions of Parabolic PDE ( with S-J Sheu), Applied Math and Optimization, 13 (1985).

F 5. A Stochastic Control Approach to Some Large Deviations Problems, Proc. Conference on Recent Advances in Dynamic Programming, Rome, March 1984. Springer Lecture Notes in Math. No. 1119, 52-66.

F 6. An Optimal Stochastic Production Planning Problem with Randomly Fluctuating Demand, (with S.P. Sethi and H.M. Soner), SIAM J. on Control and Optimization, 25 (1987). 1494-1502.

F 7. Asymptotic Series and the Method of Vanishing Viscosity ( with P.E. Souganidis), Indiana Univ. Math. J. 35 (1986) 425-447.

F 8. Convex Duality Approach to the Optimal Control of Diffusions (with D. Vermes), SIAM J on Control and Optimiz to appear in 1989.

F 9. On the Existence of the Dominant Eigenvalue and its Application to the Large Deviation Properties of an Ergodic Markov Process ( with S. J. Sheu and H.M. Soner), 22 (1987) 187-199.

F 10. Value Functions for Two-Player, Zero-Sum Stochastic Differential Games (with P.E. Souganidis), Indiana Univ. Math J, to appear in 1989.

F 11. Asymptotic Expansions for Markov Processes with Levy Generators (with H. M. Soner), submitted to Applied Math. and Optimiz, 19 (1989) 203-223.

F 12. A Regular Pertubation Expansion in Nonlinear Filtering, (with R.W. McGwier), Proc. 22nd IEEE Conf. on Decision and Control, December 1983, pp. 82-83.

F 13. A PDE Approach to Asymptotic Estimates for Optimal Exit Probabilities, (with P.E. Souganidis), Springer Lecture Notes in Control and Information Science. Proc. IFIP Conf., Marseille, March 1984.

F 14. Asymptotic Series for Solutions to the Dynamic Programming Equation for Diffusions with Small Noise, (with P.E. Souganidis) Proc. 24th IEEE Conf. on Decision and Control, Ft. Lauderdale, Florida, December 11-13, Vol. 1, 1985.

F 15. A Stochastic Production Planning Problem with Random Demand, (with H.M. Soner) Proc. 24th IEEE Conf. on Decision and Control, Vol. 1, Ft. Lauderdale, Florida, December 11-13, 1985.

F 16. Generalized Solutions in the Optimal Control of Diffusions (with D. Vermes) Proc. IMA Workshop, June 1986, IMA Vols. in Math. and Applic. No. 10, Springer-Verlag, 1987, 119-127.

F 17. Two-Player, Zero-Sum Stochastic Differential Games (with P.E. Souganidis) Proc. of Conf. in honor of J.L. Lions, June 1988.

H.J. Kushner

Kushner's research covered a wide range of topics in stochastic systems theory and applied probability. These include: large deviations with communications applications, stochastic approximations ( convergence theorems, large deirations estimates), adaptive filters, distributed parameter stochastic systems, wide band noise approximations, Monte Carlo methods, distributed and communicating stochastic approximation algorithms, and singular stochastic control.

The work on large deviations and applications is reported in [K1] [K3] [K12] [K16] [K25]. Typical communications applications arise in models with rapidly varying noise inputs, for slowly adapting digital systems, and for tracking systems with small noise effects. Among the accomplishments is a "quick simulation"technique, based on a change of probability measure technique. This method relies on the numerical solution of a first-order nonlinear

partial differential equation, connected with the action functional being minimized to obtain the large deviation rate.

The papers on stochastic approximation and recursive algorithms [K2] [K4] [K7] [K13] [K18] [K19] [K28] provide limit theorems and large deviations estimates under conditions on dynamics and noise which are broad enough to fit most current applications in control and communication theory. The global behavior of stochastic approximations was studied by Monte Carlo methods in [K15], and asymptotic properties of asymptotic filters via weak convergence methods in [K5].

Kushner's work on distributed parameter stochastic systems, described by stochastic partial differential equations, is reported in [K6] [K9] [K11]. This work is concerned with such questions as stability and near-stationarity for systems with wide-band noise inputs, and with nonlinear filtering applications.

Further work on wide-band noise approximations and filtering in stochastic control is reported in [K10] [K14] [K17] [K22] [K24]. In the part of this work concerned with stochastic control, the problem is to find nearly-optimal control laws for a wide-band noise driven system, based on optimal control laws for an idealized white-noise driven system ( for which there is an extensive theory.)

Recent work by Kushner on stochastic approximation [K20] [K21] concerns distributed and communicating systems, in which processers are located at physically distinct sites.

Routing and flow control problems for queues under heavy traffic can be analyzed using methods of singular stochastic control. Such problems, and related problems for wide-band noise driven singular stochastic control systems were analyzed in [K26] [K27].

Research Publications - H.J. Kushner

K 1. Robustness and approximation of escape times and large deviations estimates for systems with small noise effects, SIAM J. on Appl. Math., February 1984.

K 2. An invariant measure approach to the convergence of stochastic approximations with state dependent noise, SIAM J. on Control and Optimization, January 1984.

K 3. Asymptotic behavior of stochastic approximation and large deviations. IEEE Trans. on Automatic Control, AC - 29, 1984, pp 984-990.

K 4. Stochastic approximation in Hilbert space, SIAM J. on Control and Optimization, 23, 1985, p. 774.

K 5. Weak convergence and asymptotic properties of adaptive filters with constant gains. IEEE Trans. on Information Theory, IT 30, 1984, p 177-182.

K 6. Limits for parabolic partial differential equations with wide band stochastic coefficients, and an application to filtering theory, Stochastics, 14, 1985, pp 115-148.

K 7. Stochastic approximations via large deviations: asymptotic properties, (with P. DuPuis) SIAM, J on Control and Optimization, September 1985.

K 8. Approximating multiple Ito integrals with "band-limited" processes, Stochastics, 14, 1985, pp 85-114.

K 9. Weak convergence approximations for partial differential equations with stochastic coefficients, Stochastics, 14, 1985, pp 115-148.

K 10. Some non-linear filtering problems with wide bandwidth observation noise; Proc. Conf. on Second Control, IEEE Control Systems Society, New York, 1984.

K 11. Asymptotic properties, stability and "near" stationanity of parabolic partial differential equations with wide bandwidth inputs, Stochastics, 16, 1986, pp 111-136.

K 12. Large deviations estimates for systems with small noise effects, and applications to stochastic systems theory, (with P. DuPuis) SIAM J. on Control and Optimization, 24, 1986, pp 979-1008.

K 13. The theory of large deviations and asymptotic analysis of recursive algorithms and stochastic approximation (with P. DuPuis), in Advances in Statistical Signal Processing, ed. by H.V. Poor, JAI Press.

K 14. Approximate and limit results for nonlinear filters with wide bandwidth observation noise, Stochastics, 16, 1986, pp 65-96.

K 15. Asymptotic global behavior for stochastic approximations with slowly decreasing gain: Global optimization via Monte-Carlo, in SIAM J. on Appl. Math., 47, 1987, 169-185.

K 16. Stochastic systems with small noise, analysis and simulation, a phase locked loop example, SIAM J. Appl. Math., 47, 1987, 643-661.

K 17. Nearly optimal state feedback controls for stochastic systems with wide-band noise disturbances, (with W. Runggaldier) SIAM J. Control and Optimization, 25, 1987, 298-315.

K 18. Constrained stochastic approximation by the theory of large deviations, 'Robbins Symposium Volume', published by Wiley.

K 19. Asymptotic behavior of constrained stochastic approximations via the theory of large deviations, Z. Wahrscheinlichkeitsteorie, 75,1987, p 224-244.

K 20. Asymptotic properties of distributed and communicating stochastic approximation algorithms (with G. Yin), in SIAM J. Control and Optimization, 25, 1987; 1266-1290.

K 21. Stochastic approximation algorithms for parallel and distributed processing ( with G. Yin), Stochastics, 22, 1987, p 219-250.

K 22. Filtering and control for wide bandwith noise driven systems IEEE Trans. on Automatic control, T-AC87, 1987, p 123-133.

K 23. Direct averaging and perturbed test function methods for weak convergence, in Stochastic Optimization, Lecture Notes in Control and Inf. Sciences 81, Springer.

K 24. Almost optimal controls for wide band noise driven systems, Institute of Math. and Appl. (Minnesota) Volume 10, pub. by Springer, 1987 ( W. Fleming, P.L. Lions, ed.).

K 25. Upper bounds for large deviations for non-smooth stochastic difference eqns., LCDS Dept. 87-8 (Feb. 87) sub. to App. Prob.

K 26. Nearly optimal singular controls for wideband noise driven systems, (with K. M. Ramachandran), SIAM J. on Control and Optimization [26], p 569-591, 1988.

K 27. Optimal and approximately optimal control policies for queues in heavy traffic ( with K.M. Ramachandran) to appear SIAM J. on Control and Optimization.

K 28. Stochastic approximation and large deviations: General results for w. p. 1 convergences, (with P. DuPuis) to appear SIAM J. on Control and Optimization, LCDS Dept. 87-21, 1987.

P.E. Souganidis

Souganidis worked in the area of first and second order nonlinear partial differential equations, where a number of results concerning approximations with error estimates, singular perturbations and representation formulae were found. He also worked on questions of stability of special solutions of equations in fluids, as well as the structure of solutions of equations describing flows through porus media (implosion of waves).

A main part of this work was concerned with viscosity solutions of nonlinear partial differential equations and related problems in control and singular perturbations. Some of this work was joint with Fleming, and was mentioned above. Other results concern regularity and represention formulas for solutions of first order PDE's with no convexity hypothesis on the nonlinearity [S2] [S4] and blow-up of solutions [S5]. For second order PDE's, a result about uniqueness of viscosity solutions was obtained [S12] [S20].

In the realm of singular perturbations, the results included a PDE approach to large deviations [S6] [S13] [S16] [S17]. Other applications of similar PDE techniques were to nonlinear parabolic equations depending on a small parameter (the eikonal approximation) including porus medium equations [S8] [S10] [S18]. Other work is concerned with wave-front propagation problems in reaction-diffusion systems [S15] and instability of traveling waves for PDE's of KdV type.

Research Publications P.E. Souganidis

S 1. PDE - viscosity solution approach to some problems of large deviations (with W.H. Fleming), *Ann. di Scuola Norm. Sup.*, Serie IV, 13 (1986) 171-192.

S 2. Differential games, optimal control and directional derivatives of viscosity solutions of Bellman's and Isaacs' equations, II (with P.L. Lions), SIAM J. *on Control and Optimization* 24 (1986) 1086-1089.

S 3. A remark about viscosity solutions on the boundary, *Proc. of AMS* 96 (1986) 323-330.

S 4. A regularity result for viscosity solutions of Hamilton - Jacobi equations in one space dimension ( with R. Jensen), *Trans. AMS* 301 (1987) 137-147.

S 5. Blow-up of solutions of Hamilton-Jacobi equations (with A. Friedman), *Comm. in PDE* II (1986) 397-443.

S 6. Asymptotic series and the method of vanishing viscosity ( with W.H. Fleming), *Ind. U. Math. J. 35* (1986) 425-448.

S 7. Instability results concerning equations of KdV type (with J.L. Bona and W.A. Strauss), *Proc. R. Soc. Lond. A 411* (1987) 395-412.

S 8. The relation between the porous-medium and eikonal equations in several space dimensions (with P.L.Lions and J.L. Vazquez), *Rev. Mat. Iberoamericana* [to appear].

S 9. Maximal solutions and universal bounds for some partial differential equations of evolutions (with M.G. Crandall and P.L. Lions), *Arch Rat. Mech. Anal.* [to appear].

S 10. A PDE approach to geometric optics for certain semilinear parabolic equations (with L.C. Evans), Ind. *Math. J.* [to appear].

S 11. Value functions for two-player, zero-sum stochastic differential games (with W.H. Fleming), *Ind U. Math J.* to appear in 1989.

S 12. A uniqueness result for viscosity solutions of second-order fully nonlinear partial differential equations ( with R. Jensen and P.L. Lions), *Proc. of AMS* 102 (1988) 975-978.

S 13. A PDE approach to certain large deviation problems for systems of parabolic equations (with L.C. Evans), *Anal. Nonl. Ann. H. Poincare* [to appear].

S 14. Difference schemes for quasi-nonlinear evolution equations (with M.G. Crandall) *J. Nonl. Analysis, Theory, Methods and Applications* [to appear].

S 15. Wave front propagation for reaction-diffusion systems of PDE (with G. Barles and L.C. Evans) [to appear].

Proceedings and Reports

S 16. A PDE Approach to Some Large Deviations Problems (with W.H. Fleming), *in Nonlinear Systems of PDE in Applied Mathematics*, AMS Lectures in Applied Mathematics 23, AMS-SIAM Summer Seminar 1984.

S 17. Asymptotic Series for Solutions to the Dynamic Programming Equation for Diffusions with Small Noise (with W.H. Fleming), *in 24th IEEE Conference on Decision and Control.* December 1985.

S 18. The Relation Between the Porous Medium and Eikonal Equations, *in Proceedings of International Conference on nonlinear Partial Diff. Equations.* February 1986, L'Aquila, Italy [to appear].

S 19. Recent Developments in the Theory of Hamilton-Jacobi Equations,*Dynamics of Infinite Dimensional Systems* (S.N. Chow and J.K. Hale eds), NATO ASI Series, Springer-Verlag, New York, 1987.

S 20. Viscosity Solutions of Second-Order Equations, Stochastic control and Stochastic Differential Games, (with P.L. Lions), *in Stochastic Differential Systems, Stochastic Control Theory and Applications.* (W.H. Fleming and P.L. Lions eds), The IMA Volumes in Math. and Appl. No. 10, Springer Verlag, New York 1987.

S 21. Two player, zero-sum stochastic differential games (with W.H. Fleming) *in Analyse mathematique et applications. Contributions en l'honneur J.L. Lions.* Gauthier-Villars Paris 1988.

## OTHER ACTIVITIES AND HONORS

Each of the four investigators (Dafermos, Fleming, Kushner, and Souganidis) were frequently invited to speak at scientific meetings and universities worldwide, including several major/plenary addresses. Kushner completed the book *Approximation and Weak convergence Methods for Random Processes with Applications to Stochastic Systems Theory*, MIT Press, 1984. Fleming wrote a monograph *Controlled Markov Processes and Viscosity Solutions of Nonlinear Evolution Equations*, Accademia Nazionale dei Lincei, 1986. This monograph was based on Fleming's Lezioni Fermiane lecture series at the Scuola Normale Superiore, Pisa. Fleming was awarded (with H. Federer) a Steele Prize by the American math. Society in 1987. Souganidis is a Presidential Young Investigator and a Sloan Fellow.